Equivariant injectivity of crossed products

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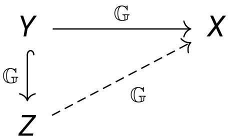
GOAL AND CONTENT

1. Introduce the notion of an action $\alpha : X \curvearrowleft \mathbb{G}$ of a locally compact quantum group on an operator space/system.
2. Study the notion of equivariant injectivity for such an operator space/system:



3. Introduce associated crossed product operator space/system $X \rtimes_\alpha \mathbb{G}$.
4. Investigate the canonical actions

$$X \rtimes_\alpha \mathbb{G} \curvearrowleft \mathbb{G}, \qquad X \rtimes_\alpha \mathbb{G} \curvearrowleft \check{\mathbb{G}}.$$

Result (1): $\check{\mathbb{G}}$-injectivity of $X \rtimes_\alpha \mathbb{G}$

Let $\alpha : X \curvearrowleft \mathbb{G}$ be a $\mathbb{G}$-operator system. The following are equivalent:

- X is $\mathbb{G}$-injective.
- $X \rtimes_\alpha \mathbb{G}$ is $\check{\mathbb{G}}$-injective and $\alpha(X) = (X \rtimes_\alpha \mathbb{G})^{\check{\mathbb{G}}}$.

Result (2): $\mathbb{G}$-injectivity of $X \rtimes_\alpha \mathbb{G}$

Let $\alpha : X \curvearrowleft \mathbb{G}$ be a $\mathbb{G}$-operator system. The following are equivalent:

- $X \rtimes_\alpha \mathbb{G}$ is $\mathbb{G}$-injective.
- $X \rtimes_\alpha \mathbb{G}$ is injective and $\mathbb{G}$ is amenable.

WHAT IS KNOWN ABOUT THESE RESULTS?

- No equivariance on the crossed product? Studied in the von Neumann algebra setting by Anantharamon-Delaroche and in the operator space setting by Hamana (classical groups).
- Bearden-Crann 21': Established (1) in the von Neumann setting under extra injectivity assumption (classical groups).
- De Ro-Hataishi 23': Established (1) for $\mathbb{G}$ a discrete quantum group.

Special cases:

- Applying (1) with $X = \mathbb{C}$, we find that $\mathbb{G}$ is amenable if and only if $L^\infty(\check{\mathbb{G}}) = \mathbb{C} \rtimes \mathbb{G}$ is $\check{\mathbb{G}}$-injective (Crann 17').
- Applying (2) with $X = L^\infty(\mathbb{G})$ and $\Delta : L^\infty(\mathbb{G}) \curvearrowleft \mathbb{G}$, we find that $B(L^2(\mathbb{G})) = L^\infty(\mathbb{G}) \rtimes_\Delta \mathbb{G}$ is $\mathbb{G}$-injective if and only if $\mathbb{G}$ is amenable.

OVERVIEW

1. Operator spaces/systems.
2. Locally compact quantum groups.
3. Equivariant operator spaces.
4. Equivariant injectivity of operator spaces.
5. Crossed products.
6. Equivariant injectivity of crossed products.
7. Application 1: Non-commutative Poisson boundary.
8. Application 2: Injective envelopes and crossed products.

OPERATOR SPACES AND FUBINI TENSOR PRODUCT

Operator space

A (concrete) operator space X is a norm-closed linear subspace of $B(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space.

Fubini tensor product

Let $X \subseteq B(\mathcal{H}), Y \subseteq B(\mathcal{K})$ be operator spaces. We define the Fubini tensor product $X \bar{\otimes} Y$ to be the set of all $z \in B(\mathcal{H} \otimes \mathcal{K})$ such that $(\omega \bar{\otimes} \mathrm{id})(z) \in Y$ for all $\omega \in B(\mathcal{H})_*$ and $(\mathrm{id} \bar{\otimes} \chi)(z) \in X$ for all $\chi \in B(\mathcal{K})_*$.

LOCALLY COMPACT QUANTUM GROUPS

- A Hopf-von Neumann algebra is a pair (M, Δ) where M is a von Neumann algebra and $\Delta : M \to M\bar{\otimes}M$ a unital, normal, faithful $*$-homomorphism such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
- A locally compact quantum group (= lcqg) $\mathbb{G}$ is a Hopf von Neumann algebra $(L^\infty(\mathbb{G}), \Delta)$ with invariant weights.
- We let $L^2(\mathbb{G})$ be a standard Hilbert space for the von Neumann algebra $L^\infty(\mathbb{G})$.
- There are fundamental multiplicative unitaries $V, W \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ such that

$$\Delta(x) = W^*(1 \otimes x)W = V(x \otimes 1)V^*, \quad x \in L^\infty(\mathbb{G}).$$

- We denote the predual of $L^\infty(\mathbb{G})$ by $L^1(\mathbb{G})$.

- We define

$$C_0(\mathbb{G}) = [(\omega \otimes \mathrm{id})(V) : \omega \in L^1(\mathbb{G})] = [(\mathrm{id} \otimes \omega)(W) : \omega \in L^1(\mathbb{G})]$$

which is a σ-weakly dense C^*-subalgebra of $L^\infty(\mathbb{G})$ with $\Delta(C_0(\mathbb{G})) \subseteq M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$.

- We define the von Neumann algebra and the coproduct

$$L^\infty(\check{\mathbb{G}}) = [(\mathrm{id} \otimes \omega)(V) \mid \omega \in L^1(\mathbb{G})]^{\sigma\text{-weak}}$$
$$\check{\Delta} : L^\infty(\check{\mathbb{G}}) \to L^\infty(\check{\mathbb{G}}) \bar{\otimes} L^\infty(\check{\mathbb{G}}) : x \mapsto V^*(1 \otimes x)V.$$

- We also need the following notations:

$$\Delta_l : B(L^2(\mathbb{G})) \to L^\infty(\mathbb{G}) \bar{\otimes} B(L^2(\mathbb{G})) : x \mapsto W^*(1 \otimes x)W,$$
$$\Delta_r : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G})) \bar{\otimes} L^\infty(\mathbb{G}) : x \mapsto V(x \otimes 1)V^*,$$
$$\check{\Delta}_r : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G})) \bar{\otimes} L^\infty(\check{\mathbb{G}}) : x \mapsto \check{V}(x \otimes 1)\check{V}^*.$$

- A lcqg $\mathbb{G}$ is called compact if $C_0(\mathbb{G})$ is unital, and we then write $C_0(\mathbb{G}) = C(\mathbb{G})$.
- A lcqg $\mathbb{G}$ is called discrete if $\check{\mathbb{G}}$ is compact.
- A lcqg $\mathbb{G}$ is called amenable if there exists a state $m : L^\infty(\mathbb{G}) \to \mathbb{C}$ such that

$$(m \otimes \mathrm{id})(\Delta(x)) = m(x)1, \quad x \in L^\infty(\mathbb{G}).$$

- A lcqg $\mathbb{G}$ is called inner amenable if there exists a state $n : L^\infty(\check{\mathbb{G}}) \to \mathbb{C}$ such that

$$(n \otimes \mathrm{id})(\Delta_r(x)) = n(x)1, \quad x \in L^\infty(\check{\mathbb{G}}).$$

EQUIVARIANT OPERATOR SPACES

$\mathbb{G}$-operator space/system (cfr. Hamana 91')

- A (right) $\mathbb{G}$-operator space is a pair (X, α) where X is an operator space and $\alpha : X \to X \bar{\otimes} L^\infty(\mathbb{G})$ is a complete isometry such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\alpha\quad} & X\bar{\otimes}L^\infty(\mathbb{G}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\otimes\mathrm{id}} \\ X\bar{\otimes}L^\infty(\mathbb{G}) & \xrightarrow[\mathrm{id}\otimes\Delta]{} & X\bar{\otimes}L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G}) \end{array}$$

  commutes.
- A $\mathbb{G}$-operator system is a $\mathbb{G}$-operator space (X, α) where X is an operator system and α is unital.

EQUIVARIANT OPERATOR SPACES

$\mathbb{G}$-dynamical von Neumann algebra

A $\mathbb{G}$-dynamical von Neumann algebra is a $\mathbb{G}$-operator system (M, α) where M is a von Neumann algebra and $\alpha : M \to M \bar{\otimes} L^\infty(\mathbb{G})$ is a normal $*$-homomorphism.

Important examples

Fix a $\mathbb{G}$-dynamical von Neumann algebra (M, α). Define the operator systems

$$X := [(\omega \otimes \mathrm{id})\alpha(m) : m \in M, \omega \in M_*] \subseteq L^\infty(\mathbb{G}),$$
$$Y := [(\mathrm{id} \otimes \omega)\alpha(m) : m \in M, \omega \in L^1(\mathbb{G})] \subseteq M.$$

Then $\Delta(X) \subseteq X \bar{\otimes} L^\infty(\mathbb{G})$ and $\alpha(Y) \subseteq Y \bar{\otimes} L^\infty(\mathbb{G})$, so (X, Δ) and (Y, α) are $\mathbb{G}$-operator systems.

$\mathbb{G}$-EQUIVARIANT MAPS

$\mathbb{G}$-equivariant map

Let (X, α) and (Y, β) be two $\mathbb{G}$-operator spaces. A completely bounded map $\phi : X \to Y$ is said to be $\mathbb{G}$-equivariant if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & Y \\ \alpha\downarrow & & \downarrow\beta \\ X\bar{\otimes}L^\infty(\mathbb{G}) & \xrightarrow[\phi\otimes\mathrm{id}]{} & Y\bar{\otimes}L^\infty(\mathbb{G}) \end{array}$$

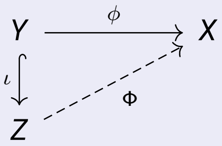
commutes. If we want to emphasize the actions, we write this as

$$\phi : (X, \alpha) \to (Y, \beta).$$

$\mathbb{G}$-EQUIVARIANT INJECTIVITY

$\mathbb{G}$-equivariant injectivity

A $\mathbb{G}$-operator space (resp. system) X is said to be $\mathbb{G}$-injective as a $\mathbb{G}$-operator space (resp. $\mathbb{G}$-operator system) if for every (resp. unital) completely isometric $\mathbb{G}$-equivariant map $\iota : Y \to Z$ and every (resp. unital) completely contractive $\mathbb{G}$-equivariant map $\phi : Y \to Z$, there exists a $\mathbb{G}$-equivariant complete contraction $\Phi : Z \to X$ such that the following diagram commutes:



$\mathbb{G}$-AMENABILITY AND RELATION WITH $\mathbb{G}$-INJECTIVITY

$\mathbb{G}$-amenability - Averaging procedure for the action

A $\mathbb{G}$-operator space (X, α) is said to be $\mathbb{G}$-amenable if there exists a $\mathbb{G}$-equivariant completely contractive conditional expectation

$$E : (X \bar{\otimes} L^{\infty}(\mathbb{G}), \mathrm{id} \otimes \Delta) \to (\alpha(X), \mathrm{id} \otimes \Delta).$$

Characterisation equivariant injectivity

Let (X, α) be a $\mathbb{G}$-operator space. The following are equivalent:

1. (X, α) is $\mathbb{G}$-injective (as a $\mathbb{G}$-operator space).
2. X is injective and (X, α) is $\mathbb{G}$-amenable.

If moreover (X, α) is a $\mathbb{G}$-operator system, then this is also equivalent with

3 (X, α) is $\mathbb{G}$-injective (as a $\mathbb{G}$-operator system).

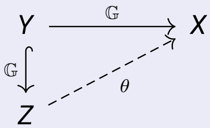
Proof. (1) $\implies$ (2)

Embed $X \subseteq B(\mathcal{H})$. We have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & X\bar{\otimes}L^\infty(\mathbb{G}) & \xrightarrow{\subseteq} & B(\mathcal{H} \otimes L^2(\mathbb{G})) \\ \downarrow{\scriptstyle \mathrm{id}_X} & & & \swarrow{\scriptstyle \exists} & \\ X & & & & \end{array}$$

Injectivity of $B(\mathcal{H} \otimes L^2(\mathbb{G}))$ implies the injectivity of X.

Proof. (2) $\implies$ (1).



By $\mathbb{G}$-amenability of (X, α), there is a $\mathbb{G}$-equivariant completely contractive map $E : (X \bar{\otimes} L^\infty(\mathbb{G}), \mathrm{id} \otimes \Delta) \to (X, \alpha)$ such that $E \circ \alpha = \mathrm{id}_X$. Consider the composition

$$Z \xrightarrow{\alpha_Z} Z \bar{\otimes} L^\infty(\mathbb{G}) \xrightarrow{\theta \otimes \mathrm{id}} X \bar{\otimes} L^\infty(\mathbb{G}) \xrightarrow{E} X.$$

It is $\mathbb{G}$-equivariant and makes the above diagram commute. □

CROSSED PRODUCTS

Fubini crossed product (cfr. Hamana 91')

Let (X, α) be a $\mathbb{G}$-operator space. We define the Fubini crossed product operator space

$$X \rtimes_\alpha \mathbb{G} = \{z \in X \bar{\otimes} B(L^2(\mathbb{G})) : (\alpha \otimes \mathrm{id})(z) = (\mathrm{id} \otimes \Delta_l)(z)\}.$$

- If (X, α) and (Y, β) are $\mathbb{G}$-operator spaces and $\phi : (X, \alpha) \to (Y, \beta)$ a $\mathbb{G}$-equivariant completely bounded map, we have

$$(\phi \otimes \mathrm{id})(X \rtimes_\alpha \mathbb{G}) \subseteq Y \rtimes_\beta \mathbb{G}.$$

- We write $\phi \rtimes \mathbb{G} := (\phi \otimes \mathrm{id})|_{X \rtimes_\alpha \mathbb{G}} : X \rtimes_\alpha \mathbb{G} \to Y \rtimes_\beta \mathbb{G}$.

Dynamics of the Fubini crossed product

We have

$$(\mathrm{id} \otimes \Delta_r)(X \rtimes_\alpha \mathbb{G}) \subseteq (X \rtimes_\alpha \mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$$
$$(\mathrm{id} \otimes \check{\Delta}_r)(X \rtimes_\alpha \mathbb{G}) \subseteq (X \rtimes_\alpha \mathbb{G}) \bar{\otimes} L^\infty(\check{\mathbb{G}}).$$

Consequently, $(X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \Delta_r)$ is a $\mathbb{G}$-operator space and $(X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r)$ is a $\check{\mathbb{G}}$-operator space.

Life is not always nice:

- It may happen that $\alpha(X) \subsetneq (X \rtimes_\alpha \mathbb{G})^{\mathrm{id}\otimes\check{\Delta}_r}$.
- The Takesaki-Takai duality $(X \rtimes_\alpha \mathbb{G}) \rtimes_{\mathrm{id}\otimes\check{\Delta}_r} \check{\mathbb{G}} \cong X \bar{\otimes} B(L^2(\mathbb{G}))$ may not hold.

The following notion lies at the core of these issues:

$\mathbb{G}$-complete operator space

A $\mathbb{G}$-operator space (X, α) is said to be $\mathbb{G}$-complete if for every inclusion $(X, \alpha) \subseteq_{\mathbb{G}} (Y, \beta)$ and every $y \in Y$ such that

$$(\mathrm{id} \otimes \omega)\beta(y) \in X$$

for all $\omega \in L^1(\mathbb{G})$, we have that $y \in X$.

Note that every $\mathbb{G}$-operator space (X, α) is $\mathbb{G}$-complete if $\mathbb{G}$ is a discrete quantum group.

Consequences of $\mathbb{G}$-completeness

Let (X, α) be a $\mathbb{G}$-operator space.

1. If (X, α) is $\mathbb{G}$-injective, it is $\mathbb{G}$-complete.
2. If (X, α) is $\mathbb{G}$-complete, then $(X \rtimes_\alpha \mathbb{G})^{\mathrm{id} \otimes \check{\Delta}_r} = \alpha(X)$.
3. If (X, α) is $\mathbb{G}$-complete, the complete isometry

$$\Phi : X \bar{\otimes} B(L^2(\mathbb{G})) \to X \bar{\otimes} B(L^2(\mathbb{G})) \bar{\otimes} B(L^2(\mathbb{G}))$$
$$\Phi(z) = V_{23}^*(\alpha \otimes \mathrm{id})(z) V_{23}, \quad z \in X \bar{\otimes} B(L^2(\mathbb{G}))$$

has image $\Phi(X) = (X \rtimes_\alpha \mathbb{G}) \rtimes_{\mathrm{id} \otimes \check{\Delta}_r} \check{\mathbb{G}}$. In other words, the Takesaki-Takai duality holds.

Remark: If $\check{\mathbb{G}}$ is amenable, then the converses of (2) and (3) are true!

Iterated crossed products

If (X, α) is a $\mathbb{G}$-operator space, then the map

$$\Phi : ((X \rtimes_\alpha \mathbb{G})\bar{\otimes} L^\infty(\check{\mathbb{G}}), \mathrm{id} \otimes \mathrm{id} \otimes \check{\Delta}) \to ((X \rtimes_\alpha \mathbb{G}) \rtimes_{\mathrm{id} \otimes \Delta_r} \mathbb{G}, \mathrm{id} \otimes \mathrm{id} \otimes \check{\Delta}_r)$$

given by $\Phi(z) = V_{23} z V_{23}^*$ is a $\check{\mathbb{G}}$-equivariant completely isometric isomorphism such that $\Phi \circ (\mathrm{id} \otimes \check{\Delta}_r) = \mathrm{id} \otimes \Delta_l$ on $X \rtimes_\alpha \mathbb{G}$.

$\check{\mathbb{G}}$-injectivity of $X \rtimes \mathbb{G}$

Let (X, α) be a $\mathbb{G}$-operator system. The following statements are equivalent:

1. (X, α) is $\mathbb{G}$-injective.
2. $(X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r)$ is $\check{\mathbb{G}}$-injective and $(X \rtimes_\alpha \mathbb{G})^{\mathrm{id} \otimes \check{\Delta}_r} = \alpha(X)$.

Proof. (1) $\implies$ (2)

Assume (X, α) is $\mathbb{G}$-injective. We may assume $X \subseteq B(\mathcal{H})$.

- There is a $\mathbb{G}$-equivariant ucp map

$$\phi : (B(\mathcal{H})\bar{\otimes}L^\infty(\mathbb{G}), \mathrm{id} \otimes \Delta) \to (X, \alpha)$$

  such that $\phi \circ \alpha = \mathrm{id}_X$.
- The composition

$$B(\mathcal{H})\bar{\otimes}B(L^2(\mathbb{G})) \cong (B(\mathcal{H})\bar{\otimes}L^\infty(\mathbb{G})) \rtimes_{\mathrm{id}\otimes\Delta} \mathbb{G} \xrightarrow{\phi \rtimes \mathbb{G}} X \rtimes_\alpha \mathbb{G}$$

  is a ucp conditional expectation.
- Conclusion: $X \rtimes_\alpha \mathbb{G}$ is injective.

Proof. (1) $\implies$ (2)

Consider a $\mathbb{G}$-equivariant ucp map

$$E : (X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \Delta_r) \to (X, \alpha), \quad E \circ \alpha = \mathrm{id}_X .$$

We can consider the $\check{\mathbb{G}}$-equivariant ucp map

$$\begin{array}{ccc} ((X \rtimes_\alpha \mathbb{G}) \bar{\otimes} L^\infty(\check{\mathbb{G}}), \mathrm{id} \otimes \mathrm{id} \otimes \check{\Delta}) & \xrightarrow{\cong} & ((X \rtimes_\alpha \mathbb{G}) \rtimes_{\mathrm{id} \otimes \Delta_r} \mathbb{G}, \mathrm{id} \otimes \mathrm{id} \otimes \check{\Delta}_r) \\ & & \big\downarrow{\scriptstyle E \rtimes \mathbb{G}} \\ & & (X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r) \end{array}$$

which implements $\check{\mathbb{G}}$-amenability of the action

$$\mathrm{id} \otimes \check{\Delta}_r : X \rtimes_\alpha \mathbb{G} \curvearrowleft \check{\mathbb{G}}.$$

Thus, $(X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r)$ is $\check{\mathbb{G}}$-injective.

Proof. (2) $\implies$ (1).

Assume that (Y, β) and (Z, γ) are $\mathbb{G}$-operator systems, $\iota : Y \to Z$ is a $\mathbb{G}$-equivariant uci map and $\phi : Y \to X$ is a $\mathbb{G}$-equivariant ucp map. We obtain a $\check{\mathbb{G}}$-equivariant unital completely positive map

$$\Theta : (Z \rtimes_\gamma \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r) \to (X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r)$$

such that the diagram on the right commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ \big\downarrow{\scriptstyle \iota} & & \\ Z & & \end{array} \qquad\qquad \begin{array}{ccc} Y \rtimes_\beta \mathbb{G} & \xrightarrow{\phi \rtimes \mathbb{G}} & X \rtimes_\alpha \mathbb{G} \\ \big\downarrow{\scriptstyle \iota \rtimes \mathbb{G}} & \overset{\Theta}{\nearrow} & \\ Z \rtimes_\gamma \mathbb{G} & & \end{array}$$

The ucp map

$$Z \cong \gamma(Z) \xrightarrow{\subseteq} (Z \rtimes_\gamma \mathbb{G})^{\mathrm{id} \otimes \check{\Delta}_r} \xrightarrow{\Theta} (X \rtimes_\alpha \mathbb{G})^{\mathrm{id} \otimes \check{\Delta}_r} = \alpha(X) \cong X$$

is $\mathbb{G}$-equivariant and makes the desired diagram commute. $\square$

Corollaries

- Let (M, α) be a $\mathbb{G}$-dynamical von Neumann algebra. Then (M, α) is $\mathbb{G}$-injective if and only if $(M \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \check{\Delta}_r)$ is $\check{\mathbb{G}}$-injective.
- $\mathbb{G}$ is amenable if and only if $L^\infty(\check{\mathbb{G}}) = \mathbb{C} \rtimes \mathbb{G}$ is $\check{\mathbb{G}}$-injective.
- Let (X, α) be a $\mathbb{G}$-operator system. The following are equivalent:
  1. $(X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \Delta_r)$ is $\mathbb{G}$-injective.
  2. $X \rtimes_\alpha \mathbb{G}$ is injective and $\mathbb{G}$ is amenable.

Proof.

Only the last point requires proof. We know that $(X \rtimes_\alpha \mathbb{G}, \mathrm{id} \otimes \Delta_r)$ is $\mathbb{G}$-injective if and only if

$$((X \rtimes_\alpha \mathbb{G}) \rtimes_{\mathrm{id} \otimes \Delta_r} \mathbb{G}, \mathrm{id} \otimes \mathrm{id} \otimes \check{\Delta}_r) \cong ((X \rtimes_\alpha \mathbb{G}) \bar{\otimes} L^\infty(\check{\mathbb{G}}), \mathrm{id} \otimes \mathrm{id} \otimes \check{\Delta})$$

is $\check{\mathbb{G}}$-injective (the fixed point property is automatic). This is equivalent with the statement that $X \rtimes_\alpha \mathbb{G}$ is injective and $(L^\infty(\check{\mathbb{G}}), \check{\Delta})$ is $\check{\mathbb{G}}$-injective, i.e. amenability of $\mathbb{G}$. $\square$

APPLICATION TO POISSON BOUNDARIES

- Let (M, α) be a $\mathbb{G}$-dynamical von Neumann algebra and consider a normal $\mathbb{G}$-equivariant ucp map $P : (M, \alpha) \to (M, \alpha)$.
- We then define the operator system of *P-harmonic elements*

$$\mathcal{H}^\infty(M, P) = \{x \in M : P(x) = x\}.$$

- It is then easily checked that $\alpha(\mathcal{H}^\infty(M, P)) \subseteq \mathcal{H}^\infty(M, P)\bar{\otimes}L^\infty(\mathbb{G})$. Thus, the restriction $\alpha_P : \mathcal{H}^\infty(M, P) \to \mathcal{H}^\infty(M, P)\bar{\otimes}L^\infty(\mathbb{G})$ turns $\mathcal{H}^\infty(M, P)$ into a $\mathbb{G}$-operator system.
- Taking a cluster point of the sequence

$$\left\{\frac{1}{n}\sum_{k=1}^{n} P^k\right\}_{n=1}^{\infty}$$

of ucp maps $M \to M$ in the point σ-weak topology, we find a $\mathbb{G}$-equivariant ucp conditional expectation

$$E : M \to \mathcal{H}^\infty(M, P).$$

Amenable actions on the Poisson boundary

Let (M,α) be a $\mathbb{G}$-dynamical von Neumann algebra.

1. If $\alpha : M \curvearrowleft \mathbb{G}$ is an amenable action, then so is $\alpha_P : \mathcal{H}^\infty(M,P) \curvearrowleft \mathbb{G}$.
2. If $\alpha : M \curvearrowleft \mathbb{G}$ is $\mathbb{G}$-injective, then so is $\alpha_P : \mathcal{H}^\infty(M,P) \curvearrowleft \mathbb{G}$. Consequently, $\mathcal{H}^\infty(M,P) \rtimes_{\alpha_P} \mathbb{G}$ is $\check{\mathbb{G}}$-injective.

Of particular importance is the case where $(M,\alpha) = (L^\infty(\mathbb{G}),\Delta)$. In that case, to any state $\mu \in C_u(\mathbb{G})^*$, we can associate a Markov operator $P_\mu : (L^\infty(\mathbb{G}),\Delta) \to (L^\infty(\mathbb{G}),\Delta)$, which is a $\mathbb{G}$-equivariant, normal, ucp map. In that case, we write $\mathcal{H}_\mu := \mathcal{H}^\infty(L^\infty(\mathbb{G}),P_\mu)$ and $\Delta_\mu := \Delta_{P_\mu}$.

- The following are equivalent:
  1. $\check{\mathbb{G}}$ is inner amenable.
  2. The action $\Delta : L^\infty(\mathbb{G}) \curvearrowleft \mathbb{G}$ is $\mathbb{G}$-amenable.
- Taking $\mu = \epsilon \in C_u(\mathbb{G})^*$, we have $\mathcal{H}_\epsilon = L^\infty(\mathbb{G})$ and $\Delta_\epsilon = \Delta$.

Amenable actions non-commutative Poisson boundary

If $\mu \in C_u(\mathbb{G})^*$ is a state, we have:

(a) If $\check{\mathbb{G}}$ is inner amenable, then the action $\Delta_\mu : \mathcal{H}_\mu \curvearrowleft \mathbb{G}$ is amenable.

(b) If $\check{\mathbb{G}}$ is amenable, then $\Delta_\mu : \mathcal{H}_\mu \curvearrowleft \mathbb{G}$ is $\mathbb{G}$-injective. Consequently, $\mathcal{H}_\mu \rtimes_{\Delta_\mu} \mathbb{G}$ is $\check{\mathbb{G}}$-injective.

INTERESTING REMARK

An amenable lcqg with a non-amenable action

Let G be a locally compact group that is not inner amenable Then $\check{G}$ is an amenable locally compact quantum group with function algebra the right group von Neumann algebra $\mathcal{R}(G)$ and coproduct uniquely determined by $\check{\Delta}(\rho_g) = \rho_g \otimes \rho_g$ for $g \in G$. The action $\check{\Delta} : \mathcal{R}(G) \curvearrowleft \check{G}$ is not amenable.

CROSSED PRODUCTS AND INJECTIVE ENVELOPES

Let X be a $\mathbb{G}$-operator system. We let $I^1_{\mathbb{G}}(X)$ be the $\mathbb{G}$-injective envelope of X, i.e. the minimal $\mathbb{G}$-injective $\mathbb{G}$-operator system containing X $\mathbb{G}$-equivariantly (existence is non-trivial).

Compatibility crossed products and injective envelopes

Let $\mathbb{G}$ be a discrete quantum group. The following statements are equivalent:

1. $(Y, \iota : X \to Y)$ is a $\mathbb{G}$-injective envelope of X.
2. $(Y \rtimes_\beta \mathbb{G}, \iota \rtimes \mathbb{G} : X \rtimes_\alpha \mathbb{G} \to Y \rtimes_\beta \mathbb{G})$ is a $\check{\mathbb{G}}$-injective envelope of $X \rtimes_\alpha \mathbb{G}$.

In particular, $I^1_{\check{\mathbb{G}}}(X \rtimes_\alpha \mathbb{G}) = I^1_{\mathbb{G}}(X) \rtimes \mathbb{G}$ as $\check{\mathbb{G}}$-operator systems.

- Proof relies on the Takesaki-Takai duality and the existence of a normal counit $\epsilon \in \ell^1(\mathbb{G})$.

- The corresponding result for $\mathbb{G}$-C^*-operator systems was already established in previous joint work with Lucas Hataishi.
- Result also true for compact quantum groups (probably).
- If $\mathbb{G}$ is a discrete quantum group, we see that

$$I^1_{\check{\mathbb{G}}}(L^\infty(\check{\mathbb{G}})) = I^1_{\check{\mathbb{G}}}(\mathbb{C} \rtimes \mathbb{G}) \cong I^1_{\mathbb{G}}(\mathbb{C}) \rtimes \mathbb{G} = C(\partial_F \mathbb{G}) \rtimes \mathbb{G}.$$

- Open question: Do we have

$$I^1_{\check{\mathbb{G}}}(X \rtimes \mathbb{G}) = I^1_{\mathbb{G}}(X) \rtimes \mathbb{G}$$

for a co-amenable lcqg $\mathbb{G}$?

THANKS FOR YOUR ATTENTION!